# SIGNAL AND SYSTEM NORMS AND SPACES

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III Level Course 02LCPRV / 01LCPRV / 01LCPIU

**“Experimental modeling: model building from experimental data”**

# Signal and system Norms and Spaces

Let $\mathbb{X}$ be a vector space over the field $\mathbb{R}$ or $\mathbb{C}$.

A **norm on** $\mathbb{X}$ is defined as a real-valued function

$$x \longmapsto \|x\|$$

which satisfies the following properties, $\forall x, y \in \mathbb{X}$ and $\forall \alpha \in \mathbb{R}$ or $\mathbb{C}$:

$$\left.\begin{array}{ll} (i) & \|x\| \geq 0 \text{ (nonnegativity)} \\ (ii) & \|x\| = 0 \text{ if and only if } x = 0 \end{array}\right\} \quad \|x\| \text{ is a positive definite function}$$

$$(iii) \quad \|\alpha x\| = |\alpha| \, \|x\| \quad \text{(homogeneity)}$$

$$(iv) \quad \|x + y\| \leq \|x\| + \|y\| \quad \text{(triangle inequality)}$$

$\mathbb{X}$ is said to be a normed space when a norm on it is defined.

# Vector Norms

Let $\mathbb{X} = \mathbb{R}^n$ or $\mathbb{C}^n$; the Hölder norms $\ell_p$ are defined by:

$$\|x\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, \quad 1 \leq p \leq \infty$$

Specific cases of interest:

$$\begin{aligned} \|x\|_1 &:= \sum_{k=1}^{n} |x_k| \\ \|x\|_2 &:= \sqrt{\sum_{k=1}^{n} |x_k|^2} \quad \text{(Euclidean norm)} \\ \|x\|_\infty &:= \max_{1 \leq k \leq n} |x_k| \end{aligned}$$

Note that:

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1, \quad \frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty$$

# Matrix Norms

Let $\mathbb{X}$ be the set of all the matrices $m \times n$ defined on $\mathbb{R}$ or $\mathbb{C}$: $\mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$;
$\mathbb{X}$ is a vector space where different norms can be defined.

Let us focus on the class of the induced norms**:**

$$\|A\|_p := \max_{x\in\mathbb{R}^n,\ x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}, \quad A \in \mathbb{X}$$

where the following property holds, $\forall A, B \in \mathbb{X}$ such that the matrix product $AB$ makes sense:

$$\|AB\|_p \leq \|A\|_p \|B\|_p$$

Examples of induced norms:

$$\|A\|_1 := \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(maximum column sum)}$$

$$\|A\|_2 := \sqrt{\max_{1\leq i\leq n} \lambda_i \left(A^* A\right)} := \sigma_{max}\left(A\right)$$

$$\|A\|_\infty := \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(maximum row sum)}$$

The Frobenius norm is an example of non induced norm:

$$\|A\|_F := \sqrt{\text{Trace}\left(A^* A\right)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

# Norms and Spaces of discrete-time signals

Let us consider the following finite-dimensional vector spaces $\ell_p$ (with $1 \leq p \leq \infty$), given by the sequences $x = \{x_k\} : \mathbb{Z} \rightarrow \mathbb{C}$

$$\begin{aligned}
\ell_p(\mathbb{Z}_+) &:= \left\{ x = \{x_k\}_{k=0}^{\infty} : \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} < \infty \right\} \\
\ell_p(\mathbb{Z}_-) &:= \left\{ x = \{x_k\}_{k=-\infty}^{-1} : \left(\sum_{k=-\infty}^{-1} |x_k|^p\right)^{1/p} < \infty \right\} \\
\ell_p(\mathbb{Z}) &:= \left\{ x = \{x_k\}_{k=-\infty}^{\infty} : \left(\sum_{k=-\infty}^{\infty} |x_k|^p\right)^{1/p} < \infty \right\}
\end{aligned}$$

and endowed with the following norms, respectively:

$$\begin{aligned}
\|x\|_p &:= \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} \\
\|x\|_p &:= \left(\sum_{k=-\infty}^{-1} |x_k|^p\right)^{1/p} \\
\|x\|_p &:= \left(\sum_{k=-\infty}^{\infty} |x_k|^p\right)^{1/p}
\end{aligned}$$

# Norms and Spaces of continuous-time signals

Let us consider the following finite-dimensional vector space $L_p$ (with $1 \leq p \leq \infty$), given by Lebesgue integrable functions $f = f(t) : I \in \mathbb{R} \rightarrow \mathbb{C}$:

$$L_p(I) := \left\{ f : f \text{ is measurable, } \left( \int_I |f(t)|^p \, dt \right)^{1/p} < \infty \right\}$$

and endowed with the following norm:

$$\|f\|_p := \left( \int_I |f(t)|^p \, dt \right)^{1/p}$$

Main cases of interest:

| $p$ | **Discrete-time** | **Continuous-time** |
|---|---|---|
| 1 | $\|x\|_1 := \sum_k \lvert x_k \rvert$ | $\|f\|_1 := \int_I \lvert f(t) \rvert \, dt$ |
| 2 | $\|x\|_2 := \sqrt{\sum_k \lvert x_k \rvert^2}$ | $\|f\|_2 := \sqrt{\int_I \lvert f(t) \rvert^2 \, dt}$ |
| $\infty$ | $\|x\|_\infty := \sup_k \lvert x_k \rvert$ | $\|f\|_\infty := \operatorname{ess\,sup}_{t \in I} \lvert f(t) \rvert$ |

Main cases of interest:

| $p$ | **Discrete-time** | **Continuous-time** |
|---|---|---|
| 1 | $\|x\|_1 := \sum_k \|x_k\|$ | $\|f\|_1 := \int_I \|f(t)\| \, dt$ |
| 2 | $\|x\|_2 := \sqrt{\sum_k \|x_k\|^2}$ | $\|f\|_2 := \sqrt{\int_I \|f(t)\|^2 \, dt}$ |
| $\infty$ | $\|x\|_\infty := \sup_k \|x_k\|$ | $\|f\|_\infty := \operatorname{ess\,sup}_{t \in I} \|f(t)\|$ |

Applications:

$(i)$ a signal $s$ has *finite energy* if and only if $\|s\|_2 < \infty$;

$(ii)$ a signal $s$ is *magnitude-bounded* if and only if $\|s\|_\infty < \infty$;

$(iii)$ a signal $s \in L_p(\mathbb{R})$ is said to be *causal* if $s \in L_p(\mathbb{R}_+)$, while it is said to be *anticausal* if $s \in L_p(\mathbb{R}_-)$.

# Norms and Spaces of the frequency responses of discrete-time signals

Let $x = \{x_k\} : \mathbb{Z} \to \mathbb{C}$ be a discrete-time signal.

The frequency response of $x$ is defined as the discrete-time Fourier transform (DTFT)

$$X(\omega) = \hat{x}\left(e^{j\omega}\right) := \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k}$$

The normed space $\mathcal{L}_p$ (with $1 \le p \le \infty$) of the frequency responses of discrete-time signals is defined as:

$$\mathcal{L}_p([0, 2\pi]) := \left\{ X : \|X\|_p := \left( \frac{1}{2\pi} \int_0^{2\pi} |X(\omega)|^p \, d\omega \right)^{1/p} < \infty \right\}$$

# Norms and Spaces of the frequency responses of continuous-time signals

Let $x(t) : \mathbb{R} \to \mathbb{C}$ be a continuous-time signal.

The frequency response of $x$ is defined as the Fourier transform

$$X(\omega) = \hat{x}(j\omega) := \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

The normed space $\mathcal{L}_p$ (with $1 \leq p \leq \infty$) of the frequency responses of continuous-time signals is defined as:

$$\mathcal{L}_p(\mathbb{R}) := \left\{ X : \|X\|_p := \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^p \, d\omega \right)^{1/p} < \infty \right\}$$

# Hardy Spaces

$$\mathcal{H}_p(\mathbb{D}) := \left\{ \hat{h} : \hat{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k \text{ is analytic in } \mathbb{D},\ \|\hat{h}\|_p < \infty \right\}$$

where $\mathbb{D} := \{\lambda : |\lambda| < 1\}$, $\quad \|\hat{h}\|_p := \left( \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{h}\left(r \cdot e^{j\omega}\right) \right|^p d\omega \right)^{1/p}$

Main cases of interest related to frequency responses of discrete-time signals:

- $\mathcal{H}_2(\mathbb{D}) := \left\{ \hat{h} : \hat{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k \text{ is analytic in } \mathbb{D},\ \|\hat{h}\|_2 < \infty \right\}$, where
$$\|\hat{h}\|_2 := \left( \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{h}\left(r \cdot e^{j\omega}\right) \right|^2 d\omega \right)^{1/2}$$
- $\mathcal{H}_\infty(\mathbb{D}) := \left\{ \hat{h} : \hat{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k \text{ is analytic in } \mathbb{D},\ \|\hat{h}\|_\infty < \infty \right\}$, where
$$\|\hat{h}\|_\infty := \operatorname*{ess\,sup}_{|\lambda|<1} \left| \hat{h}(\lambda) \right|$$

- $\mathcal{H}_{\rho,M}(\mathbb{D}) := \left\{ \hat{h} : \hat{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k \text{ is analytic in } \mathbb{D}_\rho,\ \|\hat{h}\|_{\infty,\rho} \leq M \right\}$,
  where

$$\mathbb{D}_\rho := \{\lambda : |\lambda| < \rho\}$$

$$\|\hat{h}\|_{p,\rho} := \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{h}\left(\rho \cdot e^{j\omega}\right) \right|^p d\omega \right)^{1/p}, & 1 \leq p < \infty \\ \operatorname*{ess\,sup}_{\lambda \in \mathbb{D}_\rho} \left| \hat{h}(\lambda) \right|, & p = \infty \end{cases}$$

- $\mathcal{H}^{(1)}_{\rho=1,M}(\mathbb{D}) := \left\{ \hat{h} : \hat{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k \text{ is analityc in } \mathbb{D},\ \|\hat{h}'\|_{\infty} \leq M \right\}$,
  where

$$\hat{h}' := \frac{d\hat{h}}{d\lambda}$$

# System Norms

Let $G$ be a discrete or continuous-time, linear, time-invariant (LTI) system and let $\widehat{G}$ be the Lambda ($\widehat{G} = \sum_k g_k \lambda^k$) or Laplace transform of its impulse response $\{g_k\}$ or $g(t)$.

Let $u$ and $y = g * u$ be the input and the output of $G$, respectively. Then:

$$\|G\|_{2,2} := \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} = \|\widehat{G}\|_\infty = \begin{cases} \operatorname*{ess\,sup}_{\omega \in [0,2\pi]} \left| \hat{g}\left(e^{j\omega}\right) \right| & (D.T.) \\ \operatorname*{ess\,sup}_{\omega \in ]-\infty,\infty[} \left| \hat{g}(j\omega) \right| & (C.T.) \end{cases}$$

$$\|G\|_{\infty,\infty} := \sup_{u \neq 0} \frac{\|Gu\|_\infty}{\|u\|_\infty} = \|G\|_1 = \begin{cases} \sum_{k=-\infty}^{\infty} |g_k| & (D.T.) \\ \int_{-\infty}^{\infty} |g(t)|\, dt & (C.T.) \end{cases}$$

Applications:

- $G$ has a bounded energy amplification if and only if $\widehat{G} \in \mathcal{H}_\infty$;
- $G$ is BIBO-stable if and only if $\|G\|_1$ is bounded.

Note moreover that, if $G$ is BIBO-stable:

$$\|G\|_{2,\infty} := \sup_{u\neq 0} \frac{\|Gu\|_\infty}{\|u\|_2} = \|\widehat{G}\|_2 = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\hat{g}\left(e^{j\omega}\right)\right|^2 d\omega\right)^{1/2} & (D.T.) \\ \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{g}(j\omega)\right|^2 d\omega\right)^{1/2} & (C.T.) \end{cases}$$

$$= \|G\|_2 = \begin{cases} \left(\sum_{k=-\infty}^{\infty} |g_k|^2\right)^{1/2} & (D.T.) \\ \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^{1/2} & (C.T.) \end{cases}$$

$$\|G\|_{\infty,2} := \sup_{u\neq 0} \frac{\|Gu\|_2}{\|u\|_\infty} = \infty$$

# Banach Spaces

Let $\mathbb{X}$ be a normed vector space:

- a sequence $x = \{x_k\} \in \mathbb{X}$ is said to be convergent if

$$\exists x^* \in \mathbb{X} : \|x_k - x^*\| \longrightarrow 0 \quad \text{when } k \to \infty;$$

- a sequence $x = \{x_k\} \in \mathbb{X}$ is called a Cauchy sequence if

$$\forall \varepsilon > 0, \ \exists n > 0 : \|x_i - x_k\| \leq \varepsilon, \quad \forall i, k \geq n;$$

- $\mathbb{X}$ is said to be complete if every Cauchy sequence in $\mathbb{X}$ is convergent.

A **Banach space** is a complete normed vector space.

Examples of Banach spaces: $\mathbb{R}^n$, $\mathbb{C}^n$, $\ell_p(\mathbb{Z})$, $L_p(I)$, $\mathcal{L}_p([0, 2\pi])$, $\mathcal{L}_p(\mathbb{R})$.

# Hilbert Spaces

Let $\mathbb{X}$ be a vector space over the field $\mathbb{R}$ or $\mathbb{C}$.

An **inner (or scalar) product on $\mathbb{X}$** is defined as a complex-valued function

$$(x, y) \longmapsto \langle x, y \rangle$$

which satisfies the following properties, $\forall x, y, z \in \mathbb{X}$ and $\forall \alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$:

$$\left.\begin{array}{ll} (i) & \langle x, x \rangle \geq 0 \text{ (nonnegativity)} \\ (ii) & \langle x, x \rangle = 0 \text{ if and only if } x = 0 \end{array}\right\} \quad \langle \cdot, \cdot \rangle \text{ is a positive definite function}$$

$(iii)$ $\quad \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ (additivity)

$(iv)$ $\quad \langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ (homogeneity)

$(v)$ $\quad \langle x, y \rangle = \overline{\langle y, x \rangle}$ (symmetry)

An **Hilbert space** is defined as a complete normed vector space with an inner product that induces a norm:

$$\|x\| := \sqrt{\langle x, x\rangle}$$

Examples of Hilbert spaces:

$$\mathbb{X} = \mathbb{C}^n, \qquad \langle x, y\rangle := x^H y = \sum_{k=1}^{n} \overline{x}_k y_k$$

$$\mathbb{X} = \mathbb{C}^{m\times n}, \qquad \langle A, B\rangle := \text{Trace}\left(A^* B\right)$$

$$\mathbb{X} = \ell_2\left(\mathbb{Z}\right), \qquad \langle x, y\rangle := x^H y = \sum_{k=-\infty}^{\infty} \overline{x}_k y_k$$

$$\mathbb{X} = L_2\left(I\right), \qquad \langle f, g\rangle := \int_I \overline{f\left(t\right)} g\left(t\right) dt, \quad \text{where } I = \mathbb{R},\ \mathbb{R}_+,\ \mathbb{R}_-,\ j\mathbb{R}$$

$$\mathbb{X} = \mathcal{L}_2\left([0, 2\pi]\right), \qquad \langle X, Y\rangle := \frac{1}{2\pi}\int_0^{2\pi} \overline{X\left(\omega\right)} Y\left(\omega\right) d\omega$$