Recap about dynamic systems and models

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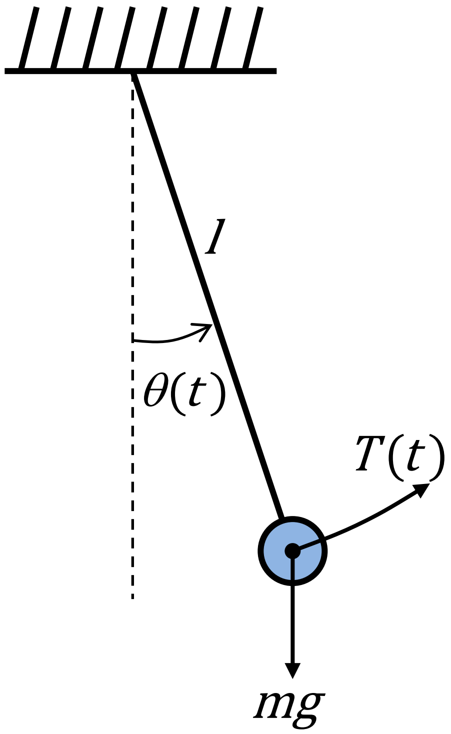
Dynamic systems - General definition

A dynamic system can be (roughly) defined as a set of interacting objects which evolve over time.

Examples:

- vehicles
- mechanical systems
- electrical circuits
- aircrafts
- spacecrafts, satellites
- stock market
- animal population
- atmosphere
- planet systems
- and so on...

Dynamic systems - A classic example: pendulum



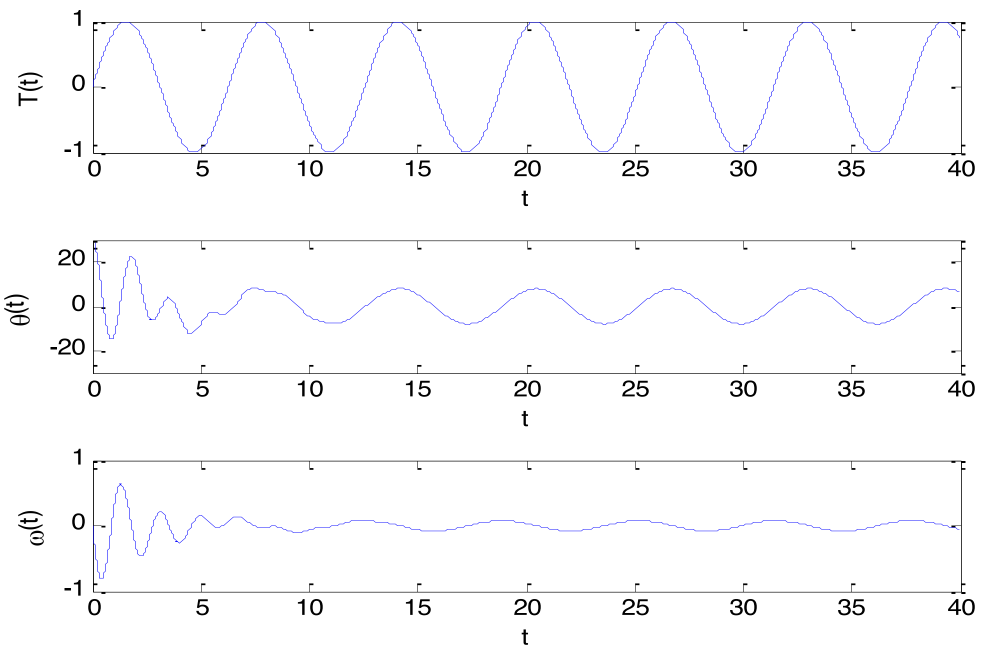
- Variables:
  - $\theta(t)$: angular position
  - $\omega(t) = \dot{\theta}(t)$: angular velocity
  - $T(t)$: applied torque.

- Parameters:
  - m : mass
  - l : length
  - g : gravity acceleration.

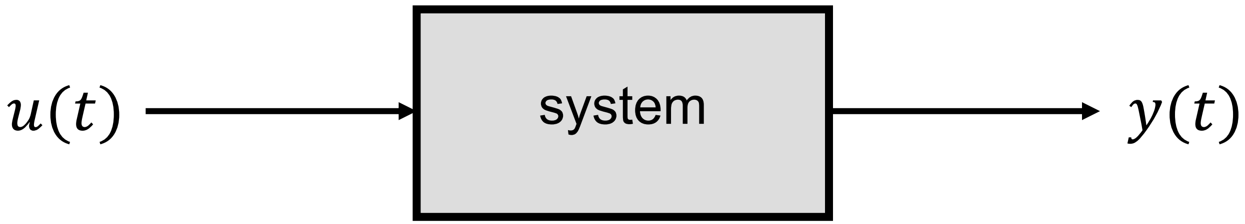
Dynamic systems - Variables and signals

- The time evolution of a dynamic system is described by quantities called variables.

- The functions which represent the time evolution of the variables are called signals.



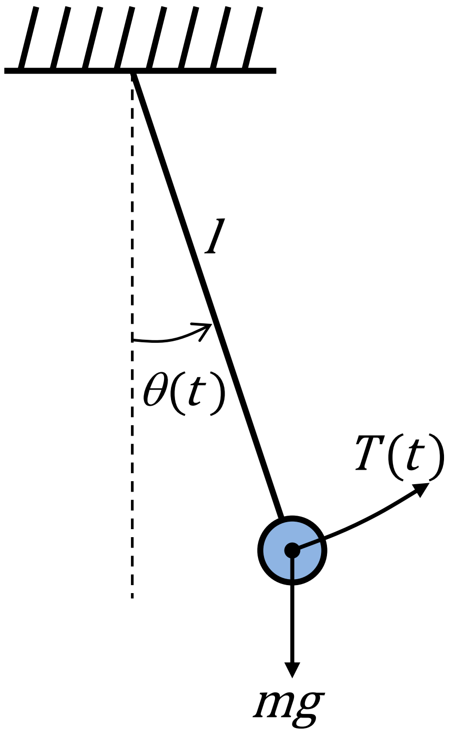
Dynamic systems - Fundamental Variables



- Fundamental variables:
  - Input $u(t)$: variables which influence the time evolution of the system (causes).
  - Output $y(t)$: measured.

- Input types:
  - Command inputs: their behavior can be chosen by the human user.
  - Disturbances: their behavior is independent on the human user; they cannot be chosen.

Dynamic systems - A classic example: pendulum



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- Parameters:
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- Input: $u(t) = T(t)$.
- Output: we can choose e.g. $y(t) = \theta(t)$.

Dynamic systems and models - A classic example: pendulum

- According to the 2nd principle of dynamics (Newton's law):

$$J\ddot{\theta}(t) = -K\sin[\theta(t)] - \beta\dot{\theta}(t) + T(t)$$

where $J = ml^2$: moment of inertia
$K = gml$: elastic constant
β : friction coefficient.

- The time evolution of the system is described by differential equations. They are a model of the system.
- The behavior of the system (more precisely, of its variables) can be predicted by integration of the differential equations.
- Integration can be performed
  - analytically (possible in particular cases)
  - numerically (always possible).

Dynamic systems and models - A classic example: pendulum

- Pendulum equation: $J\ddot{\theta}(t) = -K\sin[\theta(t)] - \beta\dot{\theta}(t) + T(t)$

- The derivative is the limit of the difference quotient:

$$\dot{\theta}(t) = \lim_{\tau\to 0}\frac{\theta(t+\tau)-\theta(t)}{\tau} \cong \frac{\theta(t+\tau)-\theta(t)}{\tau}$$

$$\ddot{\theta}(t) \cong \frac{\dot{\theta}(t+\tau)-\dot{\theta}(t)}{\tau} = \frac{\theta(t+2\tau)-2\theta(t+\tau)+\theta(t)}{\tau^2}$$

- Forward Euler discretization method:
  - Time discretized as $t = k\tau, k = 0,1,2,\ldots$, where τ is the sampling time.
  - $\theta(t) = \theta(\tau k)$, $\theta(t+\tau) = \theta(\tau k + \tau) = \theta(\tau(k+1))$, …
  - For notation simplicity, $\theta(k) = \theta(\tau k)$, $\theta(\tau(k+1)) = \theta(k+1)$, …

- Discretized pendulum equation:

$$\theta(k+2) = a_1\theta(k+1) + a_2\theta(k) + a_3\sin[\theta(k)] + bT(k)$$

where $a_1 = 2 - \frac{\tau\beta}{J}, a_2 = \frac{\tau\beta}{J} - 1, a_3 = -\frac{\tau^2 K}{J}, b = \frac{\tau^2}{J}$.

Dynamic systems and models - A classic example: pendulum

- Numerical integration by means of a Matlab script:

```
% Parameters
    m=1; l=0.8;
    J=m*l^2; K=9.81*m*l; beta=0.6;
    tau=0.01;
    a1=2-tau*beta/J; a2=-1+tau*beta/J; a3=-tau^2*K/J; b=tau^2/J;

% Initial conditions
    theta=[pi/4;pi/4];

% Time evolution
    for k=1:1998
        T(k)=sin(0.01*k);
        theta(k+2)=a1*theta(k+1)+a2*theta(k)+a3*sin(theta(k))+b*T(k);
    end
```

Dynamic systems and models – a general input-output structure

- Discretized pendulum. The following equations, obtained by a time index shift, are equivalent:

$$\theta(k+2) = a_1\theta(k+1) + a_2\theta(k) + a_3 \sin\theta(k) + bT(k)$$
$$\theta(k+1) = a_1\theta(k) + a_2\theta(k-1) + a_3 \sin\theta(k-1) + bT(k-1).$$

- In general, every discrete-time system with a finite number of states can be modeled in the following input-output form:

$$\begin{aligned} y(k+1) = f(&y(k), y(k-1), \ldots, y(k-n_a+1), \\ &u(k), u(k-1), \ldots, u(k-n_b+1)) \end{aligned}$$

  where f is a function defining the system dynamics, u is the input, y is the output, and n_a and n_b are the regressor orders.